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1996 J. Phys. A: Math. Gen. 29 4733

(http://iopscience.iop.org/0305-4470/29/15/035)

J. Phys. A: Math. Gen. **29** (1996) 4733–4739. Printed in the UK

Distribution of eigenvalues of ensembles of asymmetrically diluted Hopfield matrices

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Received 6 December 1996

Abstract. Using Grassmann variables and an analogy with two-dimensional electrostatics, we obtain the average eigenvalue distribution $\rho(\omega)$ of ensembles of $N \times N$ asymmetrically diluted Hopfield matrices in the limit $N \to \infty$. We found that in the limit of strong dilution the distribution is uniform in a circle in the complex plane.

Random matrix theory has become an active field of research in mathematics and physics in the last few decades. In physics, since the now classic work of Metha [1] on the statistical description of the energy levels of atomic nuclei, random matrices have emerged as an important tool in the study of the localization transition [2, 3], quantum chaos [4], spin glasses [5], neural networks [6], and disordered systems in general. Most of the work deals with ensembles of Hermitian or symmetric matrices whose individual properties are well known and can be exploited in more complex situations. The last few years have seen growing interest in the properties of ensembles of *asymmetric* matrices. The so-called 'circular law' for the distribution of eigenvalues of asymmetric matrices with independent entries was first obtained by Girko [7]. For example, in dynamical theories of non-symmetric spin glasses, the distribution of eigenvalues of the interaction matrix is of fundamental importance in the determination of the different phases of the system. It is found that the presence of asymmetry effects can destroy spin glass freezing [8]. In the modelling of neural networks, asymmetry of the synaptic matrix is a biologically appealing characteristic [6]. In models based on *analogue* neurons, which are somewhat more realistic than discrete ones, the properties of the eigenvalues of the synaptic matrix determine the stability of the attractors of the dynamics. In [9, 10] this analysis is carried out for symmetric matrices. The extension for asymmetric matrices is still an open and interesting problem. Asymmetry of the synaptic matrix is known to be responsible for complex dynamical behaviour in models of neural networks. For example in [11] it is shown how chaotic behaviour can appear in an asymmetric network. In [12] chaotic dynamics is found in a model neural network with analogue neurons. The eigenvalue spectrum of asymmetric matrices has also been considered in problems of learning in perceptron-like neural networks [13]. More recently, the study of eigenvalue distributions of asymmetric matrices has also been motivated by the appearance of quantum chaos in scattering processes [15]. In a recent paper Lehmann

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0305-4470/96/154733+07$19.50

et al develop a new version of the general supersymmetric method for studying disordered systems [16].

In an important contribution, Sommers *et al* [17] calculated the average density of eigenvalues $\rho(\omega)$ of $N \times N$ random asymmetric matrices in the limit $N \to \infty$, with elements J_{ij}, given by a Gaussian distribution with zero mean and correlations

$$N\langle\langle J_{ij}^2\rangle\rangle_J = 1 \qquad N\langle\langle J_{ij}J_{ji}\rangle\rangle_J = \tau. \tag{1}$$

They found that the eigenvalues are uniformly distributed inside an ellipse in the complex plane, whose semi axes depend on the degree of asymmetry of the ensemble τ. Generalizing this result, Lehmann *et al* [18] calculated the joint probability distribution of eigenvalues in Gaussian ensembles of real asymmetric matrices, recovering the elliptic law in the large N limit.

Motivated in neural network theory, in this paper we calculate the average eigenvalue distribution $\rho(\lambda)$ of an ensemble of asymmetrically diluted Hopfield matrices, whose elements are given by

$$J_{ij} = \frac{c_{ij}}{N}\sum_{\mu=1}^{p}\xi_i^\mu\xi_j^\mu \qquad i, j = 1, \ldots, N \tag{2}$$

where $\{\xi_i^\mu i = 1, \ldots, N, \mu = 1, \ldots, p\}$ represents a set of p random patterns. The ξ_i^μ are random independent variables that can take the values ± 1 with the same probability and the c_{ij} are random independent variables chosen according to the following distribution:

$$P(c_{ij}) = \gamma\delta(c_{ij} - 1) + (1 - \gamma)\delta(c_{ij}). \tag{3}$$

$0 \leqslant \gamma \leqslant 1$ measures the degree of dilution of the matrices. Note that, the c_{ij} being independent, the resulting matrices are asymmetric. $\gamma = 1$ corresponds to symmetric Hopfield matrices whose eigenvalue distribution is known [8, 18]. In [19] the spectral properties of a broad class of symmetric matrices are studied, of which Hopfield's matrices are a particular instance.

In order to obtain the distribution $\rho(\omega)$ we use an analogy with a two-dimensional electrostatic problem introduced in [17]. Let us define the Green function associated with the matrix J

$$G(\omega) = \frac{1}{N}\left\langle\!\!\left\langle \mathrm{Tr}\,\frac{1}{I\omega - J}\right\rangle\!\!\right\rangle_J \tag{4}$$

where $\omega = x + \mathrm{i}y$ is a complex variable, I the identity matrix and $\langle\langle\cdots\rangle\rangle_J$ denotes an average over the random variables ξ_i^μ. If λ_i, $i = 1, \ldots, N$ are the eigenvalues of J, then

$$\mathrm{Tr}\,\frac{1}{I\omega - J} = \sum_i^N \frac{1}{\omega - \lambda_i}. \tag{5}$$

For large N the sum can be approximated by an integral and the Green function becomes

$$G(\omega) = \int \mathrm{d}^2\lambda\,\frac{\rho(\lambda)}{\omega - \lambda} \tag{6}$$

where $\rho(\lambda)$ is the density of eigenvalues in the plane. The last equation suggests an analogy with a two-dimensional classical electrostatics problem in which $\rho(\lambda)$ represents the density of charge in the plane. It can be demonstrated [17] that an electrostatic potential Φ exists, satisfying

$$2\,\mathrm{Re}\,G = -\frac{\partial\Phi}{\partial x} \qquad -2\,\mathrm{Im}\,G = -\frac{\partial\Phi}{\partial y} \tag{7}$$

and which obeys Poisson's equation:

$$\nabla^2\Phi = -4\pi\rho. \tag{8}$$

Thus, in order to determine $\rho(\omega)$ we may calculate the potential Φ. Using that $\det(AB) = \det A \det B$ and $\det A^{\mathrm{T}} = \det A$ one can prove that a good definition for Φ can be

$$\Phi(\omega) = -1/N \langle\langle \ln \det((I\omega^* - J^{\mathrm{T}})(I\omega - J))\rangle\rangle_J \tag{9}$$

with ω^* the complex conjugate of ω and J^{T} the transpose of J. In what follows we will consider the case $N \to \infty$ and assume that in this limit the average and the ln operations commute [17]. By using a Grassmannian representation† of the determinant and adding a matrix $\epsilon\delta_{ij}$, with ϵ positive and infinitesimal in order to avoid zero eigenvalues, we get

$$\begin{aligned}\exp[-N\Phi(\omega)] = &\left\langle\!\!\left\langle \int_{-\infty}^{\infty}\left(\prod_{i=1}^{N} \mathrm{d}\eta_i^*\,\mathrm{d}\eta_i\right)\right.\right.\\ &\times \exp\left\{-\sum_{i,j,k}\eta_i^*(\omega^*\delta_{ik} - J_{ik}^{\mathrm{T}})(\omega\delta_{kj} - J_{kj})\eta_j - \epsilon\sum_i \eta_i^*\eta_i Bigr\}\right\rangle\!\!\right\rangle_J.\end{aligned} \tag{10}$$

After performing the average over the c_{ij} and over the random patterns $\{\xi_i^\mu\}$, we arrive at the following expression

$$\begin{aligned}\exp[-N\Phi(\omega)] = &\int_{-\infty}^{\infty}\left(\prod_{i=1}^{N}\mathrm{d}\eta_i\,\mathrm{d}\eta_i^*\right)\\ &\times \exp\{(\epsilon + |\omega|^2)Nq - \alpha N\ln t + \alpha\gamma(\omega+\omega^*)Nq - \alpha\gamma(1-\gamma)|\omega|^2Nq^2\}\\ &\times \int_{-\infty}^{+\infty}\left(\prod_{i=1}^{N}\mathrm{d}\chi_i^*\,\mathrm{d}\chi_i\right)\exp\Big\{-\sum_i \chi_i^*\chi_i[1+\alpha\gamma(1-\gamma)q]\\ &+\sum_i \chi_i^*\eta_i[\alpha\gamma - \alpha\gamma(1-\gamma)\omega q] + \sum_i \eta_i^*\chi_i[\alpha\gamma - \alpha\gamma(1-\gamma)\omega^* q]\Big\}\\ &\times \exp \alpha N \ln\Big\{1 - \gamma^2 qt\sum_i \chi_i^*\chi_i/N - \gamma t\Big(\sum_i\chi_i^*\eta_i + \sum_i \eta_i^*\chi_i\Big)\Big/N\\ &+\gamma^2 t\Big(\sum_i \eta_i^*\chi_i^*\sum_j \eta_j\chi_j + \sum_i\eta_i^*\chi_i\sum_j\chi_j^*\eta_j\Big)\Big/N^2\Big\}\end{aligned} \tag{11}$$

where $\alpha = p/N$ is the storage capacity parameter of the theory of Hopfield's neural networks and

$$q = \frac{1}{N}\sum_i \eta_i^*\eta_i \tag{12}$$

$$t = \frac{1}{1 - \frac{\gamma}{N}(\omega+\omega^*)\sum_i \eta_i^*\eta_i} \tag{13}$$

† For a review of Grassmann algebra see [3].

respectively. Next we define the following parameters:

$$q = \frac{1}{N}\sum_{i=1}^{N}\eta_i^*\eta_i \qquad z = \frac{1}{N}\sum_{i=1}^{N}\chi_i^*\chi_i$$
$$r = \frac{1}{N}\sum_{i=1}^{N}\eta_i^*\chi_i \qquad r^* = \frac{1}{N}\sum_{i=1}^{N}\chi_i^*\eta_i \tag{14}$$
$$s = \frac{1}{N}\sum_{i=1}^{N}\eta_i^*\chi_i^* \qquad s^* = \frac{1}{N}\sum_{i=1}^{N}\eta_i\chi_i$$

and introduce them into (11) by using delta functions. After integrating over the Grassmann variables we get

$$\begin{aligned}\exp[-N\Phi(\omega)] = {}& \left(\frac{N}{2\pi}\right)^4 \int_{-\infty}^{\infty} \mathrm{d}q\,\mathrm{d}Q\,\mathrm{d}z\,\mathrm{d}Z\,\mathrm{d}r\,\mathrm{d}R\,\mathrm{d}r^*\,\mathrm{d}R^* \\ & \times \exp N\bigg\{qQ + zZ + rR + r^*R^* + [\epsilon + |\omega|^2 + \alpha\gamma(\omega+\omega^*)]q \\ & +\alpha\ln[1-\gamma(\omega+\omega^*)q] + \ln(RR^* - ZQ) - z + \alpha\gamma(r+r^*) \\ & -\alpha\gamma(1-\gamma)[|\omega|^2q^2 + zq + \omega r^* q + \omega^* r q] \\ & +\alpha\ln\left[1 - \frac{\gamma^2 qz}{1-\gamma(\omega+\omega^*)q} - \frac{\gamma(r+r^*)}{1-\gamma(\omega+\omega^*)q} + \frac{\gamma^2 r r^*}{1-\gamma(\omega+\omega^*)q}\right]\bigg\}.\end{aligned} \tag{15}$$

In the large N limit this multiple integral can be evaluated by the saddle point method. Up to now the calculation is exact for arbitrary γ. Since the resulting saddle-point equations are difficult to solve analytically, in this work we present the results for the strong dilution limit ($\gamma \ll 1$). Expanding the exponent in powers of γ and keeping terms up to $\mathrm{O}(\gamma)$ we obtain, after some calculations:

$$\exp[-N\Phi(\omega)] \propto \int_{-\infty}^{+\infty} \mathrm{d}q\; \exp -N[\ln|q| - (\epsilon + |\omega|^2)q + \alpha\gamma|\omega|^2 q^2 - \ln(1+\alpha\gamma q)]. \tag{16}$$

After a change of variables $\sigma = 1/q$ we arrive at the following saddle-point equation:

$$\frac{\epsilon}{\sigma^2} = \frac{1}{\sigma+\alpha\gamma} - \frac{x^2}{(\sigma+\alpha\gamma)^2} - \frac{y^2}{(\sigma+\alpha\gamma)^2}. \tag{17}$$

Expanding ϵ in powers of σ, the solution of the saddle-point equation in the limit $\epsilon \to 0$ is $\sigma = 0$ inside the circle $x^2 + y^2 = \alpha\gamma$. In this region $G(\omega) = \omega^*/(\alpha\gamma)$ (non-analytic) and $\nabla^2\Phi = -4/(\alpha\gamma)$. This implies that the density of eigenvalues is uniform inside a circle of radius $\sqrt{\alpha\gamma}$ in the complex plane. Outside the circle the solution to (17) becomes $\sigma = x^2 + y^2 - \alpha\gamma$, the Green function is $G(\omega) = 1/\omega$ (analytic), and the density $\rho = 0$. The density of eigenvalues in the whole complex plane is:

$$\rho(\omega) = \begin{cases} 1/\pi\alpha\gamma & \text{if } x^2 + y^2 \leqslant \alpha\gamma \\ 0 & \text{otherwise.}\end{cases} \tag{18}$$

It is important to note that $\langle\langle J_{ij}J_{ji}\rangle\rangle_J \propto \gamma^2$ and consequently, in this limit of strong dilution, the matrix elements become effectively uncorrelated and we obtain a 'circular law' in accordance with Girko's results [7]. Our result can also be compared with the similar result of [17] for the case $\tau = 0$, i.e. a Gaussian ensemble of completely asymmetric random matrices. It is expected that this circle deforms into an ellipse as the asymmetry parameter γ increases and permits the appearance of random correlations between the patterns.

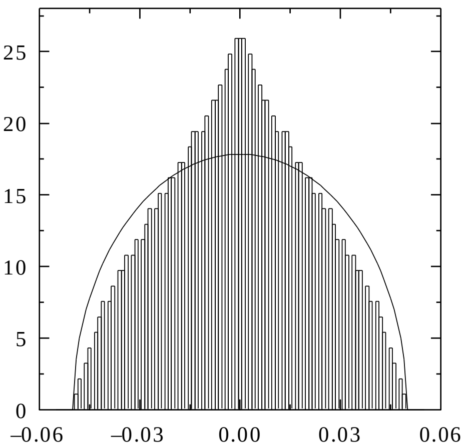

Figure 1. Projection of the eigenvalue distribution on the real axis. The full curve correspond to the analytical results and the histogram to the numerical diagonalization performed with $\gamma = 0.01$, $\alpha = 0.25$ and $N = 512$ and averaged over 20 realizations of the matrices.

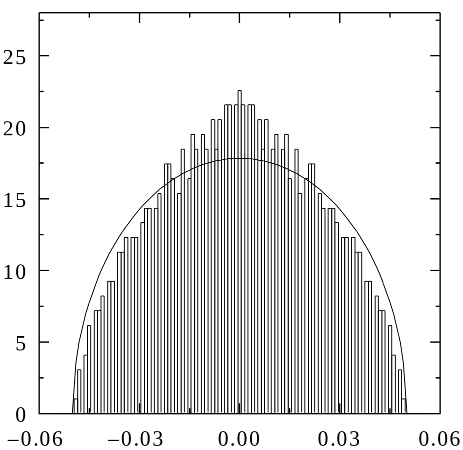

Figure 2. The same as figure 1 with $N = 1024$ and averaged over 10 realizations of the matrices.

In figures 1–3 we show the results of numerical diagonalization of sets of $N \times N$ matrices, with linear sizes N ranging from 512 to 2048 for $\alpha = 0.25$ and $\gamma = 0.01$. The figures show the projection of the distribution of complex eigenvalues in the real axis. The full curves represent the analytical solution:

$$\begin{aligned}\rho_x &= \int \rho(x, y)\,\mathrm{d}y \\ &= \frac{2}{\pi\alpha\gamma}(\alpha\gamma - x^2)^{1/2} \qquad |x| \leqslant \sqrt{\alpha\gamma}.\end{aligned} \tag{19}$$

We found that the numerical results present a peak at the origin that becomes smaller as the size N increases. Assuming that it is a finite size effect, and that the weight of the peak might be uniformly distributed on the whole support of the distribution, we renormalized the distributions. After renormalizing the numerical data the agreement with the analytic curves becomes very good as the size increases. Figure 4 shows the dependence of the peak at the origin with the system size. We have fitted the data at the origin with an exponential

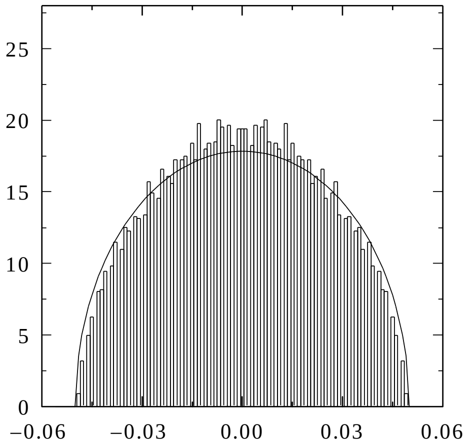

Figure 3. The same as the previous figures with $N = 2048$ averaged over eight realizations of the matrices.

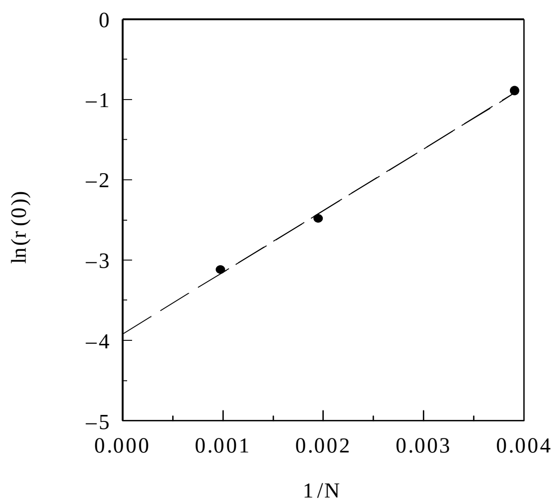

Figure 4. Finite size scaling of the peak at the origin in the complex plane $\ln\rho(0)$ versus $1/N$.

function in $1/N$, $\rho_x(0) = a\exp(b/N)$. The extrapolation to $N \to \infty$ coincides with an error of 10% with the analytic result at the origin.

To conclude, in this paper we have obtained analytically the distribution of eigenvalues of an ensemble of asymmetrically diluted Hopfield matrices in the limit of strong dilution. This limit is of particular importance because it represents a whole class of problems in which the dynamics can be solved analytically due to the absence of correlations between different sites in the network (see for example [20] and references therein). The eigenvalues are uniformly distributed inside a circle in the complex plane. Our results are supported by numerical diagonalization of the ensemble considered. Although we presented only the results for the strong dilution limit, the saddle-point equations are valid for any amount of dilution and their general solution would be welcome.

Acknowledgments

We wish to thank Professor Luca Peliti for valuable discussions and the 'Centro Nacional de Supercomputação da Universidade Federal do Rio Grande do Sul' for permitting us the use of the Cray Y-MP2E/232 where the numerical part of the work was done.

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